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Investing with cryptocurrencies - A liquidity constrained investment approach

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Cryptocurrencies have left the dark side of the finance universe and become an object of study for asset and portfolio management. Since they have a low liquidity compared to traditional assets, one needs to take into account liquidity issues when one puts them into the same portfolio. We propose use a LIquidity Bounded Risk-return Optimization (LIBRO) approach, which is a combination of the Markowitz framework under the liquidity constraints. The results show that cryptocurrencies add value to a portfolio and the optimization approach is even able to increase the return of a portfolio and lower the volatility risk.

The codes used to obtain the results in this paper are available via `www.quantlet.de` .

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1 Introduction

With the emergence of cryptocurrencies, not just a new kind of currencies and transaction networks arose, also a new kind of investment products. The cryptocurrency market shows a strong gain over the last years, which can be inferred from CRIX, developed by Trimborn and Härdle (2016) and visualized on hu.berlin/crix. The index indicates a gain of the market of 1200% over the last 1.5 years, which makes it attractive for investors. A natural question is why an investor should engage in such a risky market given the observable volatility effects.

Elendner et al. (2017) found cryptocurrencies to have a low linear dependency with each other. They also discovered the top 10 cryptocurrencies by market capitalization to have a low linear dependency with traditional assets. Since cryptocurrencies are uncorrelated with each other and traditional assets, they are interesting for investors due to the diversification effect. A first econometric analysis of CRIX has been presented by Chen et al. (2017), giving also an outlook on option pricing for CRIX. Brière et al. (2015) and Eisl et al. (2015) added Bitcoin to a portfolio of traditional assets and found an enhanced portfolio in terms of risk-return. Since alternative cryptocurrencies (alt coins, cryptos other than Bitcoin), have favorable properties too, we are aiming on constructing a portfolio consisting of traditional assets and several cryptocurrencies.

When investing with cryptocurrencies, one is confronted with a higher volatility pattern than for traditional assets, see Figure 1. Markowitz (1952) developed a method - accounting for diverging variance and covariance - in terms of a minimum variance portfolio. The approach though has drawbacks since the resulting portfolio often has extreme positive and negative weights. This may result from a single dominant factor in the covariance matrix, Green and Hollifield (1992). In an empirical study, Jagannathan and Ma (2003) find nonnegativity constraints on the weights to have an equal effect to removing the effect of a single dominant eigenvalue from the covariance matrix. Fan et al. (2012) provide theoretical insights into their findings and find constraining the weights from taking extreme positions to be more effective than nonnegativity constraints.

A further problem arises due to the low liquidity of the crypto market. In Figure

2, we make a comparison of "liquidity" measured by average daily trading amount of cryptocurrencies and S&P 500 component stocks. It is obvious that the average daily trading amount of crypto currencies are all lower than the 25% quantile of S&P 500 stocks. If we want to include cryptocurrencies and stocks into a same portfolio, we need to avoid giving cryptocurrencies a too big weight since this will induce a severe liquidity problem on adjusting the position as fast as possible. For example, if we hold a long position on an asset, which equals to twice its average daily trading amount, then it is expected to take about two days to clear this position, following the same pace of the market. However, this may result in missing the trading opportunity.

Due to the outlined challenges and the advantage from investing with cryptocurrencies, we are aiming on developing a portfolio optimization method which accounts for volatility risk and low liquidity. We call it LIBRO - LIquidity Bounded Risk-return Optimization, which is a combination of the classic Markowitz portfolio formation method and an additional restriction, which prevents big weights on low liquidity assets. The results of considering portfolios of cryptocurrencies and equities show a better Sharpe-Ratio under the liquidity constraints. Especially in the Portuguese and German market, a good performance under the liquidity constraints is observed.

The paper is organized as follows. In section 2 the Markowitz portfolio is reviewed and it is shown how the constraints increase the volatility risk of the estimated portfolio compared to an estimated unconstrained portfolio under the true covariance matrix. The size of the effect is shown in a simulation study in section 5. Section 3 introduces the liquidity constraints and section 4 gives an in-sample and out-of-sample application with 3 different stock markets - USA, Germany, Portugal - and cryptocurrencies. The results are summarized in section 6.

2 Constrained Minimum Variance portfolios

Markowitz (1952) introduced the theory of optimizing weights such that the variance of the portfolio is minimized according to a certain target return. Consider the Minimum Variance portfolio: variance is minimized without a given target return. When the variance

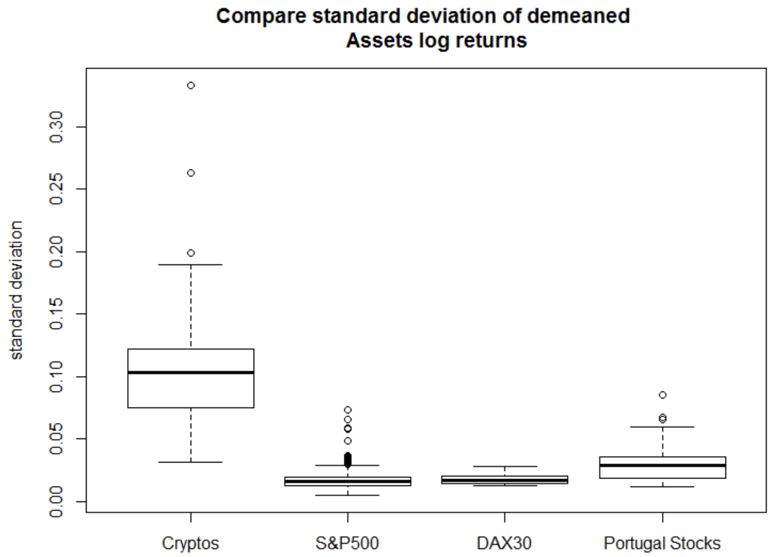

Figure 1: Crypto currencies have higher volatilities than stocks, highlighting the importance of risk management when investing on them

LIBRObox1

serves as a risk measure, this translates into risk minimization. Consider now p assets and let $\hat{\Sigma}$ be the estimated covariance matrix of the respective assets. Then the Minimum Variance portfolio is defined as, Härdle and Simar (2015):

$$\begin{aligned} &\min_w \; w^\top \hat{\Sigma} w \\ &\text{s.t.} \;\; \mathbf{1}_p^\top w = 1 \end{aligned} \tag{1}$$

where $w = (w_1, w_2, \cdots, w_p)^\top$ is the weight on assets, $\mathbf{1}_p$ is a $(p \times 1)$ matrix with all elements equals 1.

A constrained Minimum Variance portfolio is defined as,

$$\begin{aligned} &\min \; w^\top \hat{\Sigma} w \\ &\text{s.t.} \;\; \mathbf{1}_p^\top w = 1, \; ||w||_1 \leq c, \; |w| \leq a \end{aligned} \tag{2}$$

The c constraint controls the amount of shortselling, $c \in [1, \infty)$. The vector of constraints

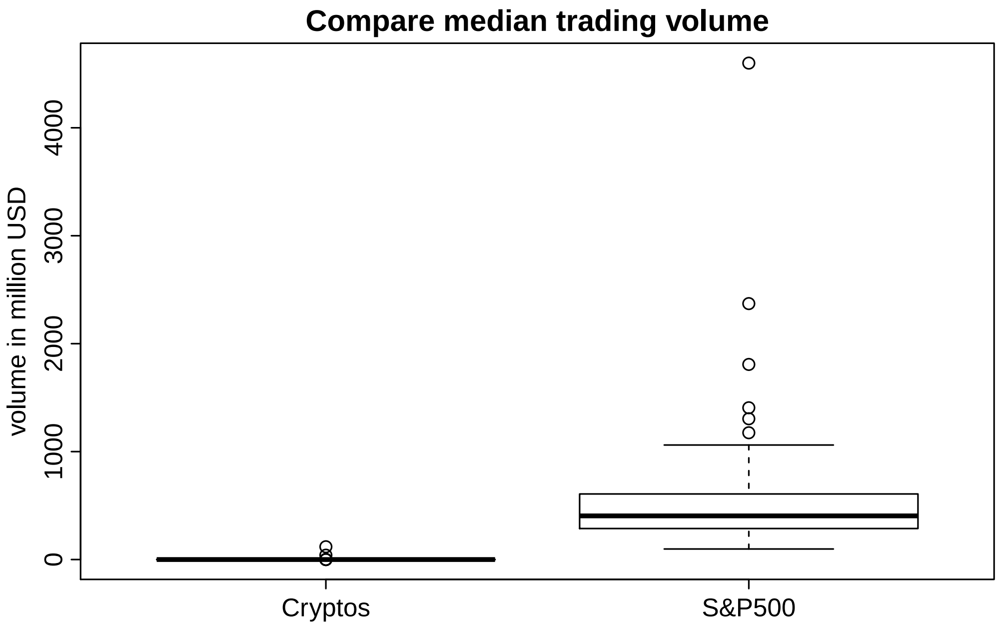

Figure 2: The figure shows the boxplot of median trading volume (measured in US dollar) of all crypto currencies and S&P 100 components, using the sample between 2014-04-01 and 2017-03-20. It is clear that crypto currencies have much lower daily trading volumes than S&P 100 component stocks.
LIBRObox2

$a = (a_1, \ldots, a_p)^\top$ with $a_i \in [0, \infty)$ for all $i = \{1, \ldots, p\}$ is a $p \times 1$ can be given (or estimated) upfront. Fan et al. (2012) showed how the risk of the estimated portfolio is influenced by the choice of c while $a_i = \infty$ for all a_i. Introducing bounds into the optimizing function on the individual weights, causes the risk to behave differently. First, the Lemma of Fan et al. (2012) is introduced before the extension to the present situation of the weights being individually restricted is given. Define the two risk functions

$$R(w) = w^\top \Sigma w \quad R_n(w) = w^\top \widehat{\Sigma} w$$

and let

$$w_{opt,a} = \underset{w^\top 1=1, ||w||_1 \le c, |w| \le a}{\arg\min} R(w) \tag{3}$$

$$\widehat{w}_{opt,\hat{a}} = \underset{w^\top 1=1, ||w||_1 \le c, |w| \le \hat{a}}{\arg\min} R_n(w)$$

$$b_n = ||\widehat{\Sigma} - \Sigma||_\infty$$

where $||\cdot||_\infty$ is the supremum norm. Fan et al. (2012) set $a = \widehat{a} = \infty$, so they worked without restrictions on the individual weights. They prove Lemma 1, but with the definitions from (3) it can be adapted.

Lemma 1. *Let $b_n = ||\widehat{\Sigma} - \Sigma||_\infty$. Then, it follows (without any conditions)*

$$
\begin{aligned}
|R(w_{opt,a}) - R_n(\widehat{w}_{opt,a})| &\leq b_n c^2 \\
|R(\widehat{w}_{opt,a}) - R_n(\widehat{w}_{opt,a})| &\leq b_n c^2 \\
|R(\widehat{w}_{opt,a}) - R(w_{opt,a})| &\leq 2 b_n c^2.
\end{aligned}
$$

Note that the Lemma holds for a being a vector of binding and non-binding restrictions. The Lemma indicates the theoretical minimum risk and the actual risk are approximately the same, when c is of moderate size and the estimation of $\widehat{\Sigma}$ is accurate. We are aiming on extending the inequalities of Lemma 1 to the case where $a \leq \infty$, and needs estimating; where $\widehat{a}$ denotes the estimated a. The intention is understanding how the restrictions influence the difference in risk. Due to the missing opportunity of short-selling in the cryptocurrency market, we are dealing just with the situation where $c = 1$. This changes $|w| \leq a$, $|\widehat{w}| \leq \hat{a}$ to $w \leq a$, $\widehat{w} \leq \hat{a}$. Let $\epsilon = \widehat{a} - a$ denote the estimation error of a. Further, besides $w_{opt,a}$ and $\widehat{w}_{opt,\widehat{a}}$ defined in (3), we have:

$$
\begin{aligned}
w_{opt,\widehat{a}} &= \underset{w^\top 1 = 1, ||w||_1 \leq c, |w| \leq \widehat{a}}{\arg\min} R(w) \\
\widehat{w}_{opt,a} &= \underset{w^\top 1 = 1, ||w||_1 \leq c, |w| \leq a}{\arg\min} R_n(w)
\end{aligned}
$$

According to the constraints, we have $w_{opt,a}, \widehat{w}_{opt,a} \leq a$ and $w_{opt,\widehat{a}}, \widehat{w}_{opt,\widehat{a}} \leq \hat{a}$. To facilitate later deduction, define

$$
\begin{aligned}
a &= w_{opt,a} + \delta_{1,w} = \widehat{w}_{opt,a} + \delta_{1,\widehat{w}}, \quad \delta_{1,w}, \delta_{1,\widehat{w}} \text{ slack variables} \\
\widehat{a} &= w_{opt,\widehat{a}} + \delta_{2,w} = \widehat{w}_{opt,\widehat{a}} + \delta_{2,\widehat{w}}, \quad \delta_{2,w}, \delta_{2,\widehat{w}} \text{ slack variables} \\
w_{opt,\widehat{a}} &= w_{opt,a} + \delta_{1,w} - \delta_{2,w} + \epsilon \\
\widehat{w}_{opt,\widehat{a}} &= \widehat{w}_{opt,a} + \delta_{1,\widehat{w}} - \delta_{2,\widehat{w}} + \epsilon
\end{aligned}
$$

The following Theorem 1 shows the added risk due to the introduction and estimation of the boundaries on the individual weights. It allows to study the effect of a false estimation of the restrictions in case of different sizes of the weights.

Theorem 1. *Let $b_n = ||\widehat{\Sigma} - \Sigma||_\infty$, a and $\widehat{a}$ be vectors of restriction, then follows*

$$
\begin{aligned}
|R(w_{opt,a}) - R_n(\widehat{w}_{opt,\widehat{a}})| &\leq b_n + |-2\widehat{w}_{opt,a}^\top \widehat{\Sigma} d_{\widehat{w}} - 2\widehat{w}_{opt,a}^\top \widehat{\Sigma}\epsilon - 2d_{\widehat{w}}^\top \widehat{\Sigma}\epsilon - d_{\widehat{w}}^\top \widehat{\Sigma} d_{\widehat{w}} - \epsilon^\top \widehat{\Sigma}\epsilon| \\
|R(\widehat{w}_{opt,a}) - R_n(\widehat{w}_{opt,\widehat{a}})| &\leq b_n + |-2\widehat{w}_{opt,a}^\top \widehat{\Sigma} d_{\widehat{w}} - 2\widehat{w}_{opt,a}^\top \widehat{\Sigma}\epsilon - 2d_{\widehat{w}}^\top \widehat{\Sigma}\epsilon - d_{\widehat{w}}^\top \widehat{\Sigma} d_{\widehat{w}} - \epsilon^\top \widehat{\Sigma}\epsilon| \\
|R(\widehat{w}_{opt,\widehat{a}}) - R(w_{opt,a})| &\leq 2b_n + |2w_{opt,a}^\top \Sigma d_w + 2w_{opt,a}^\top \Sigma\epsilon + 2d_w^\top \Sigma\epsilon + d_w^\top \Sigma d_w + \epsilon^\top \Sigma\epsilon \\
&\quad - 2\widehat{w}_{opt,a}^\top \widehat{\Sigma} d_{\widehat{w}} - 2\widehat{w}_{opt,a}^\top \widehat{\Sigma}\epsilon - 2d_{\widehat{w}}^\top \widehat{\Sigma}\epsilon - d_{\widehat{w}}^\top \widehat{\Sigma} d_{\widehat{w}} - \epsilon^\top \widehat{\Sigma}\epsilon|
\end{aligned}
$$

where $d_w = \delta_{1,w} - \delta_{2,w}$ and $d_{\widehat{w}} = \delta_{1,\widehat{w}} - \delta_{2,\widehat{w}}$. For the proof, see the Appendix 7.

Theorem 1 lets us bound the theoretical risk R and the empirical risk R_n in the following 4 scenarios.

1. $\widehat{w}_{opt,a}, w_{opt,a} < a, \widehat{w}_{opt,\widehat{a}} < \widehat{a}$

$$
\begin{aligned}
|R(\widehat{w}_{opt,a}) - R_n(\widehat{w}_{opt,\widehat{a}})| &\leq b_n \\
|R(w_{opt,a}) - R_n(\widehat{w}_{opt,\widehat{a}})| &\leq b_n \\
|R(\widehat{w}_{opt,\widehat{a}}) - R(w_{opt,a})| &\leq 2b_n
\end{aligned}
$$

2. $\widehat{w}_{opt,a} = w_{opt,a} = a, \widehat{w}_{opt,\widehat{a}} < \widehat{a}, d_w = -\delta_{2,w}, d_{\widehat{w}} = -\delta_{2,\widehat{w}}$

$$
\begin{aligned}
|R(\widehat{w}_{opt,a}) - R_n(\widehat{w}_{opt,\widehat{a}})| &\leq b_n + |-2\widehat{w}_{opt,a}^\top \widehat{\Sigma}(d_{\widehat{w}} + \epsilon) - (2d_{\widehat{w}} + \epsilon)^\top \widehat{\Sigma}\epsilon - d_{\widehat{w}}^\top \widehat{\Sigma} d_{\widehat{w}}| \\
|R(w_{opt,a}) - R_n(\widehat{w}_{opt,\widehat{a}})| &\leq b_n + |-2\widehat{w}_{opt,a}^\top \widehat{\Sigma}(d_{\widehat{w}} + \epsilon) - (2d_{\widehat{w}} + \epsilon)^\top \widehat{\Sigma}\epsilon - d_{\widehat{w}}^\top \widehat{\Sigma} d_{\widehat{w}}| \\
|R(\widehat{w}_{opt,\widehat{a}}) - R(w_{opt,a})| &\leq 2b_n + |2\widehat{w}_{opt,a}^\top (\Sigma - \widehat{\Sigma}) d_w + 2\widehat{w}_{opt,a}^\top (\Sigma - \widehat{\Sigma})\epsilon \\
&\quad + 2d_w^\top (\Sigma - \widehat{\Sigma})\epsilon + d_w^\top (\Sigma - \widehat{\Sigma}) d_w + \epsilon^\top (\Sigma - \widehat{\Sigma})\epsilon|
\end{aligned}
$$

3. $\widehat{w}_{opt,a}, w_{opt,a} < a, \widehat{w}_{opt,\widehat{a}} = \widehat{a}, d_w = \delta_{1,w}, d_{\widehat{w}} = \delta_{1,\widehat{w}}$

$$
\begin{aligned}
|R(\widehat{w}_{opt,a}) - R_n(\widehat{w}_{opt,\widehat{a}})| &\leq b_n + |-2\widehat{w}_{opt,a}^\top \widehat{\Sigma}(d_{\widehat{w}} + \epsilon) - (2d_{\widehat{w}} + \epsilon)^\top \widehat{\Sigma}\epsilon - d_{\widehat{w}}^\top \widehat{\Sigma} d_{\widehat{w}}| \\
|R(w_{opt,a}) - R_n(\widehat{w}_{opt,\widehat{a}})| &\leq b_n + |-2\widehat{w}_{opt,a}^\top \widehat{\Sigma}(d_{\widehat{w}} + \epsilon) - (2d_{\widehat{w}} + \epsilon)^\top \widehat{\Sigma}\epsilon - d_{\widehat{w}}^\top \widehat{\Sigma} d_{\widehat{w}}| \\
|R(\widehat{w}_{opt,\widehat{a}}) - R(w_{opt,a})| &\leq 2b_n + |2w_{opt,a}^\top \Sigma d_w + 2w_{opt,a}^\top \Sigma\epsilon + 2d_w^\top \Sigma\epsilon + d_w^\top \Sigma d_w + \epsilon^\top \Sigma\epsilon \\
&\quad - 2\widehat{w}_{opt,a}^\top \widehat{\Sigma} d_{\widehat{w}} - 2\widehat{w}_{opt,a}^\top \widehat{\Sigma}\epsilon - 2d_{\widehat{w}}^\top \widehat{\Sigma}\epsilon - d_{\widehat{w}}^\top \widehat{\Sigma} d_{\widehat{w}} - \epsilon^\top \widehat{\Sigma}\epsilon|
\end{aligned}
$$

4. $\widehat{w}_{opt,a} = w_{opt,a} = a, \widehat{w}_{opt,\widehat{a}} = \widehat{a}, d_w = d_{\widehat{w}} = 0$

$$
\begin{aligned}
|R(\widehat{w}_{opt,a}) - R_n(\widehat{w}_{opt,\widehat{a}})| &\leq b_n + |-2\widehat{w}_{opt,a}^\top \widehat{\Sigma}\epsilon - \epsilon^\top \widehat{\Sigma}\epsilon| \\
|R(w_{opt,a}) - R_n(\widehat{w}_{opt,\widehat{a}})| &\leq b_n + |-2\widehat{w}_{opt,a}^\top \widehat{\Sigma}\epsilon - \epsilon^\top \widehat{\Sigma}\epsilon| \\
|R(\widehat{w}_{opt,\widehat{a}}) - R(w_{opt,a})| &\leq 2b_n + |2\widehat{w}_{opt,a}^\top (\Sigma - \widehat{\Sigma})\epsilon + \epsilon^\top (\Sigma - \widehat{\Sigma})\epsilon|
\end{aligned}
$$

One sees that a small value of ϵ_i will just cause the risk to increase if the corresponding covariances are high. In case of a high value of ϵ_i, it will cause the risk to strongly increase. In a simulation study the effect of a bad estimation of $\widehat{a}$ on the risk while a poor estimation of $\widehat{w}$ is shown by using the covariance matrix from the cryptocurrency and Portugal dataset. The results show a minimal increase of the risk.

3 LIBRO: LIquidity Bounded Risk-return Optimization

In this section, we will present LIBRO. As mentioned before, the cryptocurrencies have far lower daily trading amount than traditional financial assets, causing a liquidity problem to any portfolio construction. To react to this issue, one tries to avoid holding too many illiquid assets via weight constraints $|w| \leq a$, as Darolles et al. (2012).

Many different liquidity measures were proposed in the literature, which tackle either one aspect of liquidity or aim on several aspects at the same time, Wyss (2004). In the context of this research, we are interested in

1. being able to trade the assets on the reallocation date

2. being able to sell or buy between two reallocation dates, if necessary

Naturally, an asset with a higher liquidity should be able to have a higher weight in the portfolio.

Since spread data for the cryptocurrency market are not available, all liquidity measures using such information are not applicable. Instead Turnover Value as a proxy for liquidity is going to be used. Since the time period of interest for a trading action is restricted to a daily basis, daily closing data for the standard assets and cryptocurrencies are going to be used. The Turnover Value is defined to be

$$TV_t = \sum_{i=1}^{N_t} p_i \cdot q_i \tag{4}$$

with N_t the number of trades between t and $t-1$, p_i the price of trade i and q_i the number of assets traded at trade i.

Recall that, w_i, $i = 1, \cdots, N$ denote the weight on asset i, M is the total amount we are going to invest, so Mw_i is the market value one holds on asset i. Hence the a constraint on w_i is:

$$Mw_i \leq TV_i \cdot f_i, \tag{5}$$

where f_i is a factor for asset i controlling the speed the investor want to clear the current position on asset i. The value $f_i = 0.5$ means that the position can be cleared on average within half a day. Dividing both sides of (5) by M yields a bound for w_i:

$$w_i \leq \frac{TV_i \cdot f_i}{M} = \widehat{a}_i. \tag{6}$$

Due to the inability of short-selling in the crypto market, the exposure is set to $c = 1$, which produces the no short-sell constraints combined with $\mathbf{1}_p^\top w = 1$. It follows, the portfolio optimization framework we will use in this paper is:

$$\begin{aligned} &\min\ w^\top \hat{\Sigma} w \\ \text{s.t. }\ &\mathbf{1}_p^\top w = 1,\ ||w||_1 \leq 1, \\ &w \leq \frac{1}{M} \cdot \widehat{Liq} = \widehat{a}, \end{aligned} \tag{7}$$

where $\widehat{Liq} = (TV_1 \cdot f_1, \cdots, TV_N \cdot f_N)^\top$.

4 Application

In this paper, 39 cryptocurrencies, obtained from the CRIX cryptocurrencies database (hu.berlin/crix), are used to form portfolios with traditional financial assets. We list the summary statistics of their return, together with the market cap listed in Table 4. The choice of 39 cryptocurrencies out of 1032 is a compromise between the time span and the number of assets: the more cryptocurrencies we include, the less time span we can cover since many cryptocurrencies are relatively young and have a history of less than 2 years. We also restrict our selection to those cryptos that have at least a market cap larger than one million dollars, so that we will not buy out the whole market easily. According to the data, our 39 cryptos account up to 79.2% of the whole crypto market, with a sample period from 2014-04-01 to 2017-03-20. Three sets of traditional financial assets are used: S&P 100 component stocks, DAX30 component stocks and stocks listed in Portugal Stock exchange. We will form portfolios using cryptos and these three set of assets respectively, and compare the result with the portfolio formed using these three sets of financial assets only, to see whether adding cryptocurrencies into the portfolio can provide additional gain to the risk-return trade off.

4.1 In sample portfolio formation

The purpose of an in-sample analysis is threefold: first, to check whether including cryptocurrencies can really increase the risk-return trade-off of portfolio formation; second, to see whether including altcoins - cryptocurrencies other than bitcoin - are profitable; third, to see whether the introduction of liquidity constraints are necessary. For this in-sample analysis, the optimization framework (7) without the liquidity constraint is implemented. If cryptocurrencies are given a small weight, say less than 5%, in the absent of a liquidity constraint, then it is not urgent to introduce the liquidity constraints, otherwise, we need to include the liquidity constraints to make the portfolio feasible.

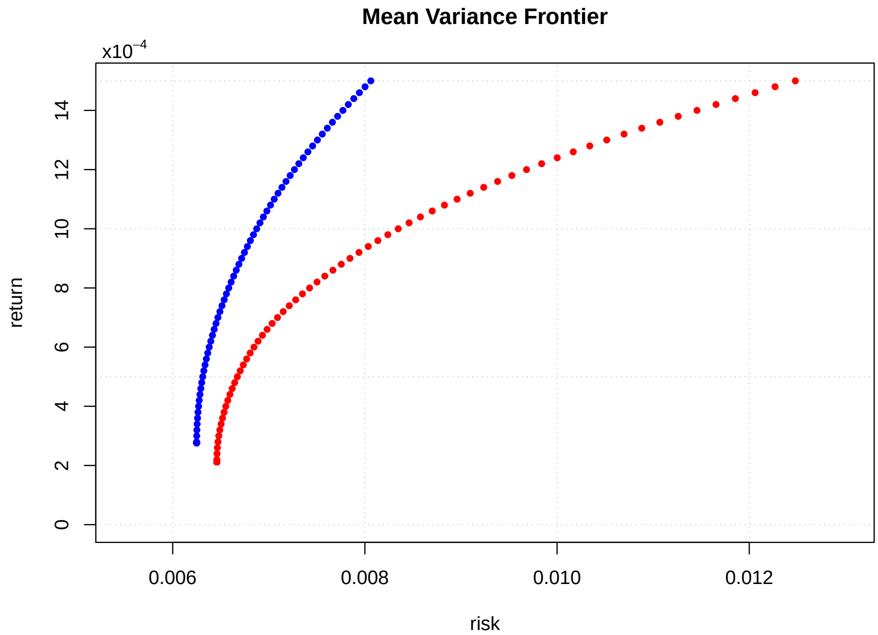

Figure 3: Mean variance frontier of S&P 100 component with/without cryptos.
LIBROinsample

Figures 3 - 5 give the mean-variance frontier of the portfolios, where the blue dots indicating those formed using traditional assets(S&P 100 component, DAX30 component

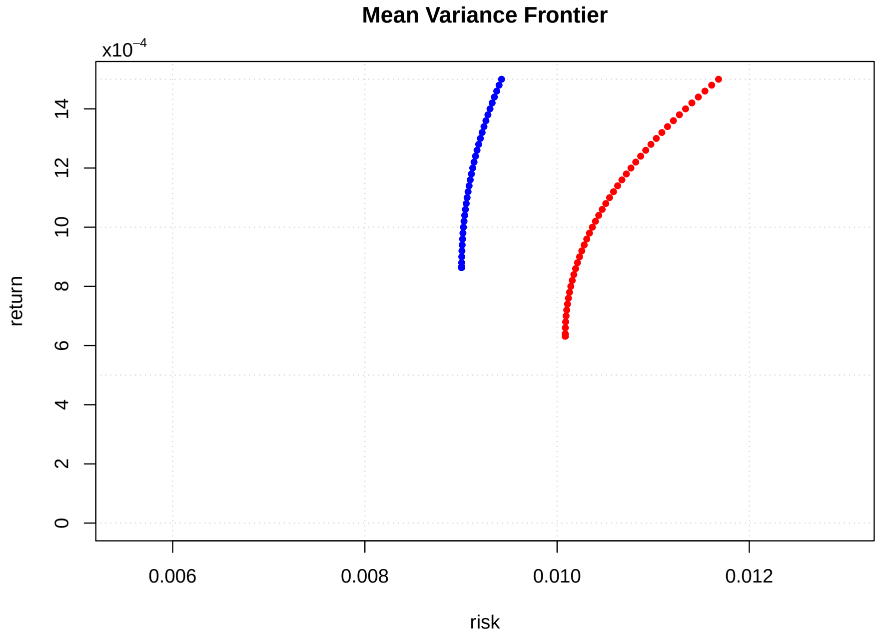

Figure 4: Mean variance frontier of DAX30 component with/without cryptos. LIBROinsample

and Portugal stocks) plus cryptocurrencies, while the red ones indicating whose formed using traditional asset only. It is obvious that all blue dots appearing on the left side of red dots, in all three figures, indicating that adding cryptocurrencies improve the risk-return trade off at every target return level. Even more, in every case, the global minimum variance point (hereinafter GMV point) with cryptos appearing on the upper-left side of the one without cryptos, indicating that adding cryptos not only decrease the risk, but also increase the return that can achieve at the GMV point.

Next we will check the weights given to cryptos, especially on altcoins. Table 2 shows the weights given to bitcoin as well as altcoins in the same portfolio formations shown in Figures 3 - 5. From the figure we can see that: first, as the target return increases, the weights given to cryptos increases, which is consistent with the observation that crypto currencies have much larger return dispersion than traditional financial assets, so they are good complements when investors want to seek high returns. Second, surprisingly, altcoins are given a relatively heavy weight compared to bitcoin: 4.6% at least, and goes up to exceeding 16.0% of the total portfolio, while, only less than 0.7% weights are given to

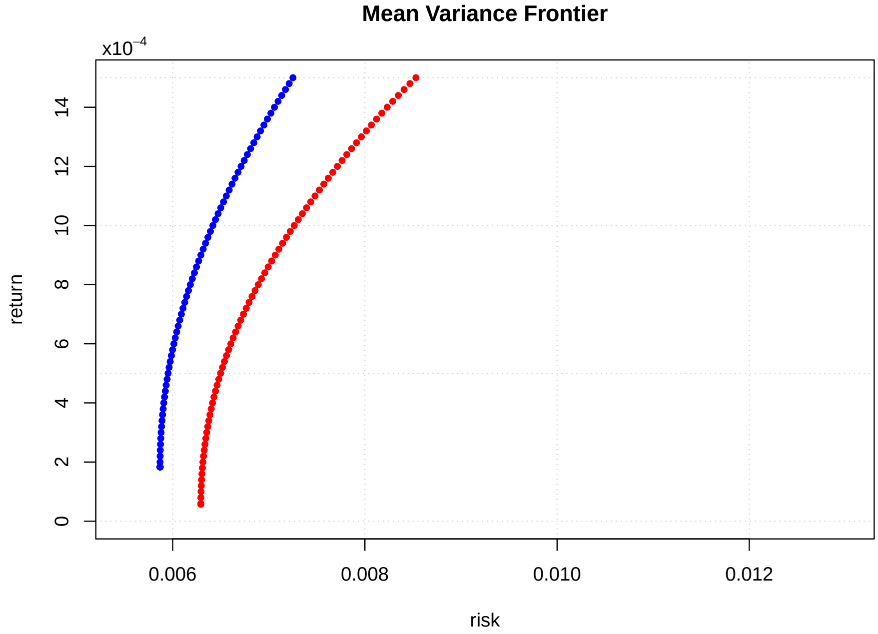

Figure 5: Mean variance frontier of Portugal stocks with/without cryptos.
LIBROinsample

bitcoin in all the three cases. This means that altcoins can be more appealing compared to the dominating bitcoin, which is exactly one of the main contribution of our paper. Third, it should be noted that the above results are not so reasonable since it gives heavy weights to altcoins, given the fact that most of them have low liquidity. So in the out-sample analysis of section 4.2, we will take the liquidity issues into consideration and include the liquidity constraints, introduced in 3.

4.2 Out of-sample portfolio formation

Let's now turn to the out sample analysis, which is illustrated in Figure 7-9. Again, we choose S&P 100 component, DAX30 component and Portugal stocks as the benchmark asset pool, and compare the performance of the portfolio with those had cryptocurrencies added. A monthly adjusted portfolio is employed, with the first trading day of each month as the adjustment day. The weight is calculated using the LIBRO method in (7), where we fix $f_i = 1$, for $i = 1, \cdots, N$, indicating that we want the portfolio adjustment done on average within one day. Our first portfolio is formed on 2016-07-01, with the weights calculated using data from 2014-04-01 to 2016-06-30. The portfolio will be kept unchanged until the last trading day of July, 2016. Then, on the first trading day of August, 2016, the portfolio will be adjusted again, using sample period from 2014-04-01 to 2016-07-31, so on and so forth. The last update will happen on March 1, 2017, with the weights calculated using sample from 2014-04-01 to 2017-03-28.

The out-of-sample performance of the portfolios are shown in Figure 7 to Figure 9, where S&P 100 component, DAX30 component and Portugal stocks are employed as base assets respectively. In each figure, we first show out-of-sample portfolio formation without liquidity constraints (letting $a = 1$ while $c = 1$ in (2)), and then show those with liquidity constraints (use LIBRO method in (7)). In each portfolio formation, we compare the cumulative return and standard deviation of portfolios with or without cryptocurrencies in the portfolio.

For S&P 100 components, when liquidity constraints are absent, adding cryptocurrencies to the portfolio shows a merely improvement on the cumulative return starting from the mid February of 2017, while in the rest of the sample, the two portfolios do not differ too much from each other. Though contradicting to the result of in-sample analysis, this is not surprising, since out-of-sample performance may be not stable when applying to a rapidly changing market like the cryptocurrencies.

However, when adding liquidity constraints to portfolio formation, things get improved, especially for the cumulative return: before October of 2016, the difference between these two portfolios does not differ from each other too much, after that, the one with cryptos starting to become outperform, and this outperforming becomes stable after mid of December of 2016. At the last day of the sample, the portfolio with cryptos ends up with 1.12% more cumulative return than the one without, which corresponds to 121,000$ more money given a total amount of 10,000,000 invested.

For the portfolio of DAX30 component stocks with/without cryptos, the situation is even better, especially after liquidity constraints are added. Before that, the out-of-sample performance of portfolios with cryptos are in fact lower performed than the one without cryptos, judging from the cumulative return as well as sharp-ratio. However, after the liquidity constraints, two main changes are witnessed: first, both portfolios have a big increase in cumulative return, compared to those without liquidity constraint. For DAX30 components only, the cumulative return is 6% higher than that without liquidity constraint; for DAX30 and cryptos, the cumulative return is 7.5% higher than that without liquidity constraint. Second, the one with cryptos becomes outperformed compared compared to the one with only stocks, and the outperform starts from mid of September, 2016. By the end of the sample, the cumulative return is 1.5% higher in the portfolio with cryptos, which corresponds to 150,000 more mony given a total amount of 10,000,000 invested.

An interesting result comes from the Portugal portfolio. When liquidity constraints are absent, the portfolio with cryptos shows lower standard deviation throughout the sample with almost the same cumulative return level. When liquidity constraints are added, the portfolio with cryptos outperform the one without about 3.4% at the end of sample period, together with lower cumulative standard deviation.

The monthly sharp-ratio is illustrated in Table 3, which gives a more directly measure of the risk-return trade off in each asset collection. For each collection, we display two different investment amount, namely 1 million dollar and 10 million dollar. The sharp ratio is calculated using a monthly growing window, with July 1 to 29 of 2016 as the starting one. For Portugal stocks, when investment amount equals 1,000,000 dollars, the portfolio with liquidity constraints will dominate the one without all the time. For DAX 30 and S&P 100 component, when investment amount is 10 million, the sharp ratio of of portfolios with liquidity constraints will dominate the one without all the time. Indicating that adding liquidity constraints can not only make the portfolio more feasible, but also improve its out-sample performance.

5 Simulation Study

Due to the best performance of cryptos in combination with the Portugal assets, the simulation study was performed based on the resulting covariance matrix from the Portugal stocks and the cryptocurrencies. So the total number of assets is $n = 80$. The covariance matrix over the entire dataset was derived and served as Σ. The second half of the time span were used to estimate $\widehat{\Sigma}$. 5 values of ϵ were chosen were $\epsilon \in \{0.01, 0.02, 0.03, 0.04, 0.05\}$. 2 groups of weights were built where

$$w \in w^{(1)} = \{w_i | 0 < w_i < 0.1\} \quad \forall i$$
$$w \in w^{(2)} = \{w_i | w_i \geq 0\} \quad \forall i.$$

The estimation error between the estimated and real weight is assumed to be 0.01, i.e. $d_w = d_{\widehat{w}} = 0.01$. The simulation study is performed for the case of $c = 1$, no shortselling. The number of restricted assets and to which group they belong is indicated in table 5. The results show, that the additional risk is small when just one of the weights is affected. But in the case of 5 weights coming from the group $w^{(1)}$, the effect becomes stronger but still stays small. For assets from group $w^{(2)}$, so a restriction on weights with higher values was imposed, the effect becomes more severe. Still, the additional volatility risk stays over all covered situations below a 10% increase. It follows, the additional risk due to the constraints is low.

The effect of the restriction on the additional risk is very low and more severe when the actual weight is 0. The higher the weight, the less intense the effect on the additional risk. So including restrictions on the weights has a minor effect on the risk.

.

	mean	STD	$Q1$	$Q2$	$Q3$	latest market cap
btc	0.0006	0.0324	-0.0095	0.0014	0.0126	1.6817e+10
dash	0.0041	0.0770	-0.0261	-0.0011	0.0274	7.2450e+08
xmr	0.0025	0.0819	-0.0351	0.0000	0.0339	3.1508e+08
xrp	-0.0003	0.0521	-0.0172	-0.0022	0.0151	2.6204e+08
ltc	-0.0008	0.0480	-0.0134	-0.0010	0.0116	2.0118e+08
maid	0.0014	0.0699	-0.0287	0.0000	0.0336	7.8960e+07
doge	-0.0008	0.0532	-0.0217	-0.0033	0.0156	2.5722e+07
sdc	0.0024	0.0969	-0.0483	-0.0005	0.0477	1.5363e+07
ppc	-0.0013	0.0598	-0.0233	-0.0036	0.0179	1.5092e+07
bts	-0.0008	0.0653	-0.0297	-0.0059	0.0218	1.4477e+07
nmc	-0.0009	0.0631	-0.0233	-0.0037	0.0157	9.4970e+06
nxt	-0.0015	0.0607	-0.0322	-0.0056	0.0228	8.9888e+06
nlg	0.0038	0.0887	-0.0412	-0.0013	0.0408	8.1045e+06
bcn	0.0005	0.0972	-0.0371	0.0002	0.0359	7.8322e+06
sjcx	0.0029	0.0815	-0.0417	0.0016	0.0405	7.7248e+06
xcp	0.0008	0.0873	-0.0475	-0.0029	0.0417	7.2179e+06
sys	0.0021	0.1017	-0.0485	-0.0036	0.0413	6.8829e+06
ioc	0.0026	0.1192	-0.0591	-0.0003	0.0628	6.7888e+06
rby	0.0040	0.1279	-0.0410	0.0032	0.0425	5.3233e+06
btcd	0.0028	0.0899	-0.0381	-0.0007	0.0392	5.2186e+06
nvc	-0.0003	0.0636	-0.0224	-0.0043	0.0149	4.9847e+06
bay	-0.0016	0.2099	-0.0607	-0.0011	0.0529	3.5906e+06
slr	0.0040	0.1281	-0.0549	0.0005	0.0547	3.4239e+06
blk	-0.0001	0.0742	-0.0311	-0.0049	0.0214	3.4184e+06
pot	0.0008	0.1014	-0.0438	-0.0014	0.0375	3.3629e+06
nav	0.0032	0.1445	-0.0598	-0.0040	0.0583	3.0020e+06
moon	0.0022	0.1157	-0.0288	0.0016	0.0289	2.2964e+06
unity	0.0000	0.3931	-0.0656	0.0023	0.0546	2.2083e+06
dgb	-0.0001	0.0907	-0.0422	-0.0066	0.0339	2.1741e+06
burst	-0.0002	0.1037	-0.0470	-0.0007	0.0407	1.9210e+06
mona	-0.0013	0.0907	-0.0296	-0.0026	0.0274	1.8514e+06
vtc	-0.0031	0.1022	-0.0468	-0.0066	0.0287	1.5926e+06
net	0.0002	0.1805	-0.0573	-0.0031	0.0423	1.5766e+06
naut	-0.0004	0.1390	-0.0733	-0.0062	0.0554	1.3959e+06
ftc	-0.0020	0.1100	-0.0454	-0.0070	0.0315	1.2850e+06
aur	-0.0031	0.0956	-0.0423	-0.0045	0.0340	1.2311e+06
cure	-0.0021	0.1447	-0.0600	-0.0037	0.0451	1.1777e+06
via	-0.0011	0.1063	-0.0520	-0.0056	0.0392	1.1007e+06
xpm	-0.0025	0.0779	-0.0339	-0.0042	0.0251	1.0490e+06

Table 1: Summary statistics of crypto currencies used in this paper. All statistics are calculated based on sample 2014-04-01 to 2017-03-20, except for "latest market cap", which is the market cap of each crypto on 2017-03-20.

$\bar{r}$	with S&P100		with DAX30		with PSI	
($\times 10^{-4}$)	BTC	ALT	BTC	ALT	BTC	ALT
0	0.007	0.046	0.000	0.137	0.000	0.068
1	0.007	0.046	0.000	0.137	0.000	0.068
2	0.007	0.046	0.000	0.137	0.000	0.068
3	0.007	0.047	0.000	0.137	-0.000	0.070
4	0.006	0.052	0.000	0.137	-0.000	0.073
5	0.006	0.056	0.000	0.137	0.000	0.075
6	0.005	0.061	0.000	0.137	0.000	0.078
7	0.004	0.068	0.000	0.137	-0.000	0.081
8	0.002	0.075	0.000	0.137	0.000	0.085
9	-0.000	0.081	0.000	0.138	0.000	0.088
10	0.000	0.087	0.000	0.142	-0.000	0.091
11	0.000	0.095	0.000	0.146	-0.000	0.095
12	-0.000	0.104	-0.000	0.150	-0.000	0.098
13	-0.000	0.113	0.000	0.155	0.000	0.102
14	-0.000	0.122	0.000	0.160	0.000	0.106
15	0.000	0.131	-0.000	0.165	0.000	0.110

Table 2: The weights on crypto currencies in in-sample analysis given different target return level. "BTC" means the weight on bitcoin, and "ALT" means the total weight on altcoins, i.e., other crypto currencies except for bitcoin.

	Portugal			DAX30			S&P100		
	1mn	10mn	no constr.	1mn	10mn	no constr.	1mn	10mn	no const.
July	0.18	0.33	0.13	0.13	0.15	0.11	0.06	0.06	0.06
Aug	0.17	0.15	0.15	0.16	0.18	0.13	-0.12	-0.12	-0.12
Sep	0.12	0.13	0.07	0.18	0.15	0.15	-0.08	-0.08	-0.08
Oct	0.13	0.10	0.08	0.15	0.15	0.13	-0.09	-0.09	-0.09
Nov	0.02	-0.02	-0.00	0.06	0.12	0.05	-0.09	-0.08	-0.09
Dec	0.06	0.04	0.02	0.10	0.16	0.09	-0.03	-0.02	-0.03
Jan	0.04	-0.00	0.03	0.07	0.14	0.07	-0.03	-0.02	-0.03
Feb	0.10	0.04	0.10	0.10	0.15	0.10	0.01	0.02	0.01
Mar	0.09	0.02	0.08	0.12	0.16	0.12	0.02	0.02	0.02

Table 3: Sharpe Ratio of portfolios with liquidity constraints for a growing window, evaluated on the end of the month, under 2 different investment amounts and one portfolio without constraints for Portugal, DAX30 and S&P100 with cryptos

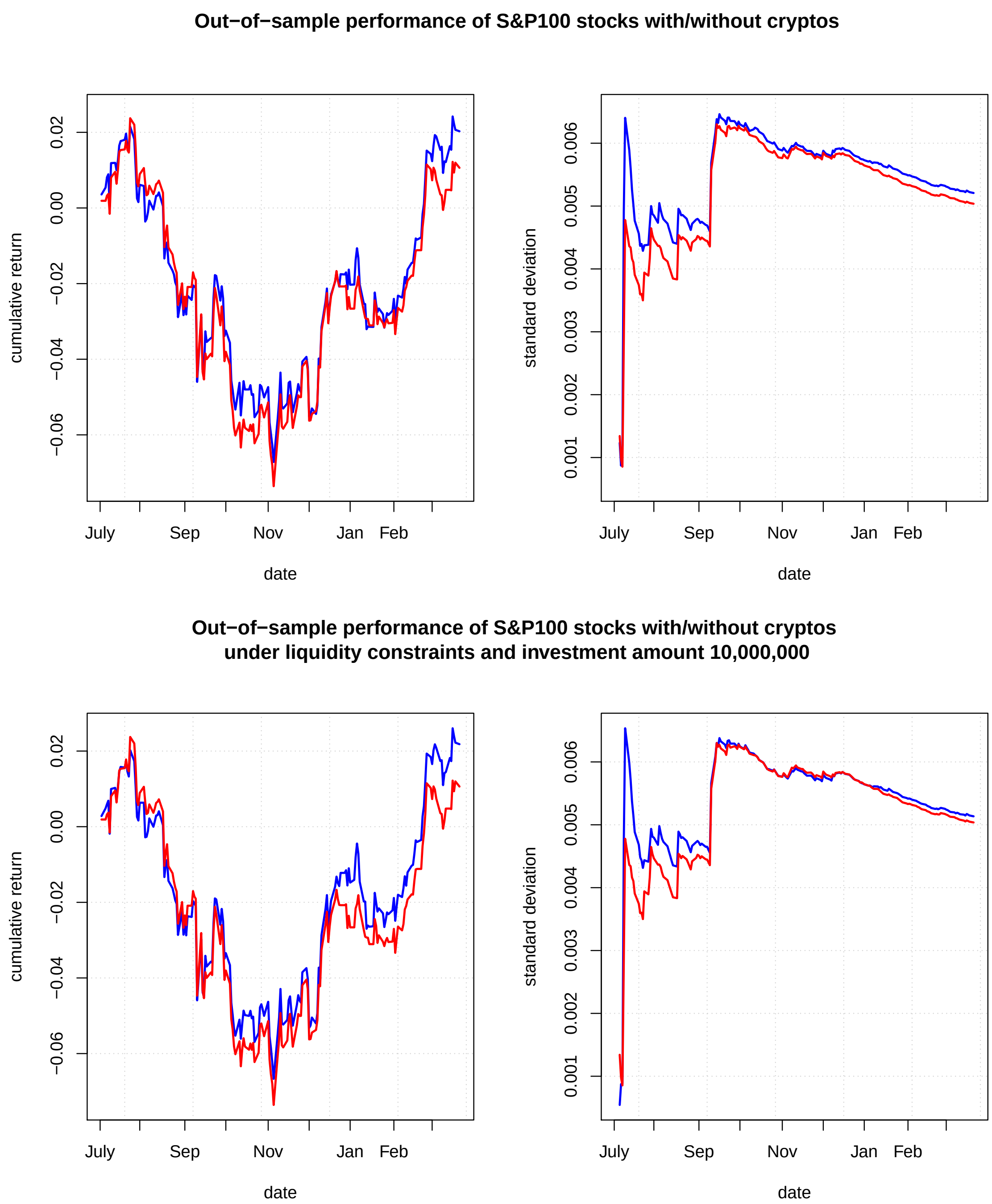

Figure 6: Cumulative return and standard deviation of S&P100 stocks with/without crypto currencies. Upper figures without liquidity constraints, below with liquidity constraints with investment amount 10,000,000.

LIBROoutsample

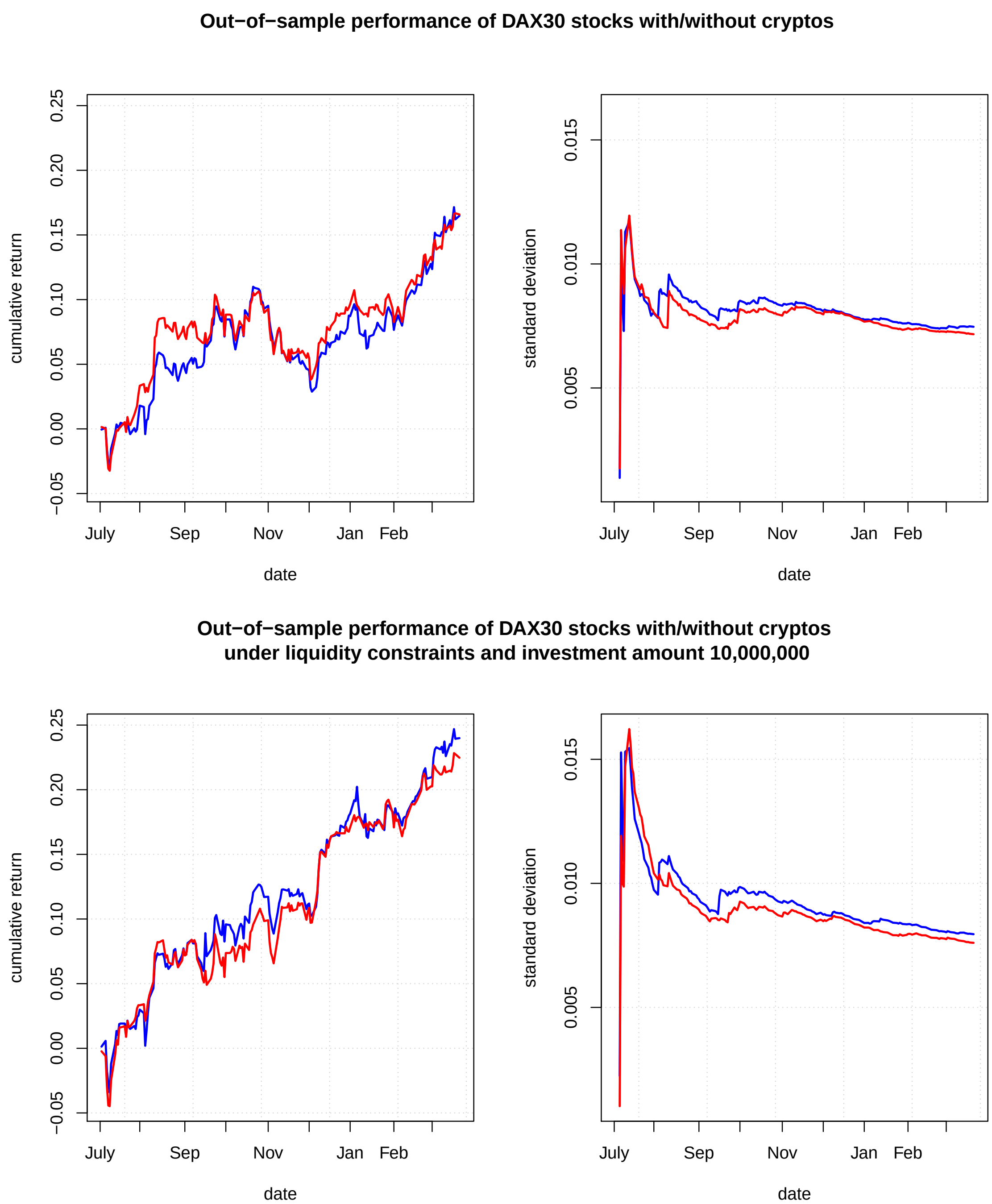

Figure 7: Cumulative return and standard deviation of DAX30 stocks with/without crypto currencies. Upper figures without liquidity constraints, below with liquidity constraints with investment amount 10,000,000.

LIBROoutsample

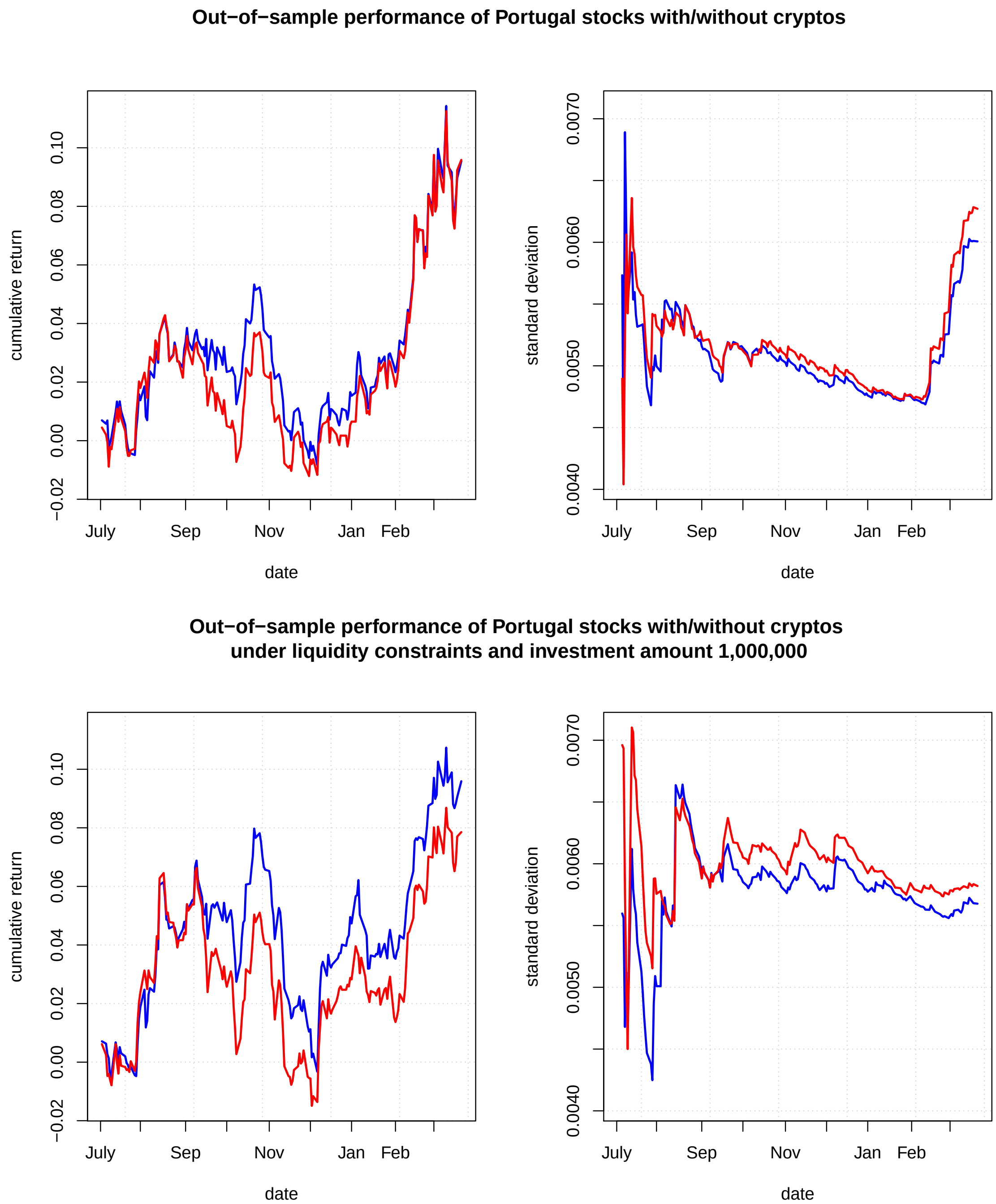

Figure 8: Cumulative return and standard deviation of Portugal stocks with/without crypto currencies. Upper figures without liquidity constraints, below with liquidity constraints with investment amount 1,000,000.

LIBROoutsample

	$\times 10^{-2}$	First 2 relations			Third relations		
		2. case	3. case	4. case	2. case	3. case	4. case
	1	0.34	0.34	0.14	0.00	0.00	0.00
	2	0.25	0.18	0.30	0.00	0.00	0.00
1 $w^{(1)}$	3	0.15	0.04	0.47	0.00	0.00	0.00
	4	0.03	0.09	0.65	0.00	0.00	0.00
	5	0.09	0.21	0.84	0.00	0.00	0.00
	1	0.92	1.53	0.12	0.00	0.00	0.00
	2	0.86	1.35	0.26	0.00	0.00	0.00
1 $w^{(2)}$	3	0.80	1.18	0.39	0.00	0.00	0.00
	4	0.74	1.01	0.54	0.00	0.00	0.00
	5	0.67	0.85	0.68	0.00	0.00	0.00
	1	1.34	2.11	0.70	0.00	0.00	0.00
	2	1.11	1.37	1.56	0.00	0.00	0.00
5 $w^{(1)}$	3	0.71	0.77	2.58	0.00	0.00	0.01
	4	0.17	0.33	3.75	0.00	0.00	0.01
	5	0.53	0.05	5.07	0.00	0.01	0.01
	1	3.64	10.13	0.52	0.01	0.00	0.00
	2	3.59	9.24	1.06	0.00	0.00	0.00
3 $w^{(2)}$	3	3.50	8.37	1.64	0.00	0.00	0.00
	4	3.39	7.54	2.24	0.00	0.00	0.00
	5	3.25	6.73	2.88	0.00	0.00	0.00

Table 4: Proportion of additional risk for the relations of Theorem 1 for 5 different epsilon and different numbers of weights of sizes: $0 < w_i < 0.1, w \in w^{(1)}$; $w_i > 0.1, w \in w^{(2)}$

LIBROsimulation

6 Conclusion

In this paper, we explore the potential gain of including crypto currencies into portfolios. On one hand, the rapid increasing cryptocurrencies are promising investment assets, while on the other hand, these cryptos are more volatile and have relatively low liquidity, so investing on them are somewhat challenging. To control the risk as well as liquidity problem, we propose a LIBRO - LIquidity Bounded Risk-return Optimization method, which extend the Markowitz framework employed in Fan et al. (2012) to contain an additional liquidity constraint. A simulation study shows that the estimation error on portfolio risk estimation can stay small after the liquidity constraints included. We do the empirical analysis by forming portfolios using S&P 100 component, DAX30 component and Portugal stocks with cryptos. It is shown that adding cryptos can improve the risk-return trade-off of portfolio formation, both in in-sample and out-sample case. Adding liquidity constraint can also improve the out-sample performance of the portfolio. Further extension of the paper can be made on try alternative portfolio formation method to improve the out-sample performance of the portfolio formation with cryptos.

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7 Appendix

With $d_{\widehat{w}} = \delta_{1,\widehat{w}} - \delta_{2,\widehat{w}}$

$$
\begin{aligned}
&|R(\widehat{w}_{opt,a}) - R_n(\widehat{w}_{opt,\hat{a}})| \\
&= |\widehat{w}_{opt,a}^\top \Sigma \widehat{w}_{opt,a} - \widehat{w}_{opt,\hat{a}}^\top \widehat{\Sigma} \widehat{w}_{opt,\hat{a}}| \\
&= |\widehat{w}_{opt,a}^\top \Sigma \widehat{w}_{opt,a} - (\widehat{w}_{opt,a} + \delta_{1,\widehat{w}} - \delta_{2,\widehat{w}} + \epsilon)^\top \widehat{\Sigma} (\widehat{w}_{opt,a} + \delta_{1,\widehat{w}} - \delta_{2,\widehat{w}} + \epsilon)| \\
&= |\widehat{w}_{opt,a}^\top \Sigma \widehat{w}_{opt,a} - (\widehat{w}_{opt,a} + d_{\widehat{w}} + \epsilon)^\top \widehat{\Sigma} (\widehat{w}_{opt,a} + d_{\widehat{w}} + \epsilon)| \\
&\leq b_n c^2 + | - 2\widehat{w}_{opt,a}^\top \widehat{\Sigma} d_{\widehat{w}} - 2\widehat{w}_{opt,a}^\top \widehat{\Sigma} \epsilon - 2 d_{\widehat{w}}^\top \widehat{\Sigma} \epsilon - d_{\widehat{w}}^\top \widehat{\Sigma} d_{\widehat{w}} - \epsilon^\top \widehat{\Sigma} \epsilon|
\end{aligned}
$$

Because $w_{opt,a}$ is optimal for Σ and $\widehat{w}_{opt,a}$ for $\widehat{\Sigma}$, it holds $R(w_{opt,a}) - R(\widehat{w}_{opt,a}) \leq 0$. From this fact it follows

$$
\begin{aligned}
&R(w_{opt,a}) - R_n(\widehat{w}_{opt,\hat{a}}) \\
&= w_{opt,a}^\top \Sigma w_{opt,a} - \widehat{w}_{opt,\hat{a}}^\top \widehat{\Sigma} \widehat{w}_{opt,\hat{a}} \\
&= w_{opt,a}^\top \Sigma w_{opt,a} - (\widehat{w}_{opt,a} + \delta_{1,\widehat{w}} - \delta_{2,\widehat{w}} + \epsilon)^\top \widehat{\Sigma} (\widehat{w}_{opt,a} + \delta_{1,\widehat{w}} - \delta_{2,\widehat{w}} + \epsilon) \\
&= w_{opt,a}^\top \Sigma w_{opt,a} - (\widehat{w}_{opt,a} + d_{\widehat{w}} + \epsilon)^\top \widehat{\Sigma} (\widehat{w}_{opt,a} + d_{\widehat{w}} + \epsilon) \\
&= R(w_{opt,a}) - R_n(\widehat{w}_{opt,a}) - 2\widehat{w}_{opt,a}^\top \widehat{\Sigma} d_{\widehat{w}} - 2\widehat{w}_{opt,a}^\top \widehat{\Sigma} \epsilon - 2 d_{\widehat{w}}^\top \widehat{\Sigma} \epsilon - d_{\widehat{w}}^\top \widehat{\Sigma} d_{\widehat{w}} - \epsilon^\top \widehat{\Sigma} \epsilon \\
&= R(w_{opt,a}) - R(\widehat{w}_{opt,a}) + R(\widehat{w}_{opt,a}) - R_n(\widehat{w}_{opt,a}) - 2\widehat{w}_{opt,a}^\top \widehat{\Sigma} d_{\widehat{w}} - 2\widehat{w}_{opt,a}^\top \widehat{\Sigma} \epsilon \\
&- 2 d_{\widehat{w}}^\top \widehat{\Sigma} \epsilon - d_{\widehat{w}}^\top \widehat{\Sigma} d_{\widehat{w}} - \epsilon^\top \widehat{\Sigma} \epsilon \\
&\leq R(\widehat{w}_{opt,a}) - R_n(\widehat{w}_{opt,a}) - 2\widehat{w}_{opt,a}^\top \widehat{\Sigma} d_{\widehat{w}} - 2\widehat{w}_{opt,a}^\top \widehat{\Sigma} \epsilon - 2 d_{\widehat{w}}^\top \widehat{\Sigma} \epsilon - d_{\widehat{w}}^\top \widehat{\Sigma} d_{\widehat{w}} - \epsilon^\top \widehat{\Sigma} \epsilon \\
&\leq b_n c^2 - 2\widehat{w}_{opt,a}^\top \widehat{\Sigma} d_{\widehat{w}} - 2\widehat{w}_{opt,a}^\top \widehat{\Sigma} \epsilon - 2 d_{\widehat{w}}^\top \widehat{\Sigma} \epsilon - d_{\widehat{w}}^\top \widehat{\Sigma} d_{\widehat{w}} - \epsilon^\top \widehat{\Sigma} \epsilon
\end{aligned}
$$

Similarly, by using $R_n(w_{opt,a}) - R_n(\widehat{w}_{opt,a}) \geq 0$, it follows

$$
\begin{aligned}
& R(w_{opt,a}) - R_n(\widehat{w}_{opt,\hat{a}}) \\
& = R(w_{opt,a}) - R_n(\widehat{w}_{opt,a}) - 2\widehat{w}_{opt,a}^\top \widehat{\Sigma} d_{\widehat{w}} - 2\widehat{w}_{opt,a}^\top \widehat{\Sigma}\epsilon - 2d_{\widehat{w}}^\top \widehat{\Sigma}\epsilon - d_{\widehat{w}}^\top \widehat{\Sigma} d_{\widehat{w}} - \epsilon^\top \widehat{\Sigma}\epsilon \\
& = R(w_{opt,a}) - R_n(w_{opt,a}) + R_n(w_{opt,a}) - R_n(\widehat{w}_{opt,a}) \\
& - 2\widehat{w}_{opt,a}^\top \widehat{\Sigma} d_{\widehat{w}} - 2\widehat{w}_{opt,a}^\top \widehat{\Sigma}\epsilon - 2d_{\widehat{w}}^\top \widehat{\Sigma}\epsilon - d_{\widehat{w}}^\top \widehat{\Sigma} d_{\widehat{w}} - \epsilon^\top \widehat{\Sigma}\epsilon \\
& \geq R(w_{opt,a}) - R_n(w_{opt,a}) - 2\widehat{w}_{opt,a}^\top \widehat{\Sigma} d_{\widehat{w}} - 2\widehat{w}_{opt,a}^\top \widehat{\Sigma}\epsilon - 2d_{\widehat{w}}^\top \widehat{\Sigma}\epsilon - d_{\widehat{w}}^\top \widehat{\Sigma} d_{\widehat{w}} - \epsilon^\top \widehat{\Sigma}\epsilon \\
& \geq -b_n c^2 - 2\widehat{w}_{opt,a}^\top \widehat{\Sigma} d_{\widehat{w}} - 2\widehat{w}_{opt,a}^\top \widehat{\Sigma}\epsilon - 2d_{\widehat{w}}^\top \widehat{\Sigma}\epsilon - d_{\widehat{w}}^\top \widehat{\Sigma} d_{\widehat{w}} - \epsilon^\top \widehat{\Sigma}\epsilon
\end{aligned}
$$

Combining the two relations, it follows

$$
|R(w_{opt,a}) - R_n(\widehat{w}_{opt,\widehat{a}})| \leq b_n c^2 + |-2\widehat{w}_{opt,a}^\top \widehat{\Sigma} d_{\widehat{w}} - 2\widehat{w}_{opt,a}^\top \widehat{\Sigma}\epsilon - 2d_{\widehat{w}}^\top \widehat{\Sigma}\epsilon - d_{\widehat{w}}^\top \widehat{\Sigma} d_{\widehat{w}} - \epsilon^\top \widehat{\Sigma}\epsilon|
$$

And for the last relation,

$$
\begin{aligned}
& R(\widehat{w}_{opt,\widehat{a}}) - R(w_{opt,a}) \\
& = R(\widehat{w}_{opt,\widehat{a}}) - R_n(\widehat{w}_{opt,\widehat{a}}) + R_n(\widehat{w}_{opt,\widehat{a}}) - R_n(w_{opt,a}) + R_n(w_{opt,a}) - R(w_{opt,a}) \\
& \leq R(\widehat{w}_{opt,\widehat{a}}) - R_n(\widehat{w}_{opt,\widehat{a}}) + R_n(w_{opt,a}) - R(w_{opt,a}) \\
& = (\widehat{w}_a + d + \epsilon)^\top \Sigma (\widehat{w}_a + d + \epsilon) - (\widehat{w}_a + d + \epsilon)^\top \widehat{\Sigma} (\widehat{w}_a + d + \epsilon) + R_n(w_{opt,a}) - R(w_{opt,a}) \\
& = R(\widehat{w}_{opt,a}) - R_n(\widehat{w}_{opt,a}) \\
& + 2w_a^\top \Sigma d_w^\top + 2w_{opt,a}\Sigma\epsilon + 2d_w \Sigma\epsilon + d_w \Sigma d_w + \epsilon\Sigma\epsilon \\
& - 2\widehat{w}_a^\top \widehat{\Sigma} d_{\widehat{w}}^\top - 2\widehat{w}_{opt,a}\widehat{\Sigma}\epsilon - 2d_{\widehat{w}}\widehat{\Sigma}\epsilon - d_{\widehat{w}}\widehat{\Sigma} d_{\widehat{w}} + \epsilon\widehat{\Sigma}\epsilon \\
& + R_n(w_{opt,a}) - R(w_{opt,a}) \\
& \leq 2c^2 |R_n(\widehat{w}_{opt,a}) - R(\widehat{w}_{opt,a})| \\
& + |2w_a^\top \Sigma d_w^\top + 2w_{opt,a}\Sigma\epsilon + 2d_w \Sigma\epsilon + d_w \Sigma d_w + \epsilon\Sigma\epsilon \\
& - 2\widehat{w}_a^\top \widehat{\Sigma} d_{\widehat{w}}^\top - 2\widehat{w}_{opt,a}\widehat{\Sigma}\epsilon - 2d_{\widehat{w}}\widehat{\Sigma}\epsilon - d_{\widehat{w}}\widehat{\Sigma} d_{\widehat{w}} + \epsilon\widehat{\Sigma}\epsilon|
\end{aligned}
$$

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